

Variants of the Erdős-Szekeres and Erdős-Hajnal Ramsey problems

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Abstract

Given integers ℓ, n , the ℓ th power of the path P_n is the ordered graph P_n^ℓ with vertex set $v_1 < v_2 < \dots < v_n$, and all edges of the form $v_i v_j$ where $|i - j| \leq \ell$. The ramsey number $r(P_n^\ell, P_n^\ell)$ is the minimum N such that every 2-coloring of $\binom{[N]}{2}$ results in a monochromatic copy of P_n^ℓ . The Erdős-Szekeres theorem says that $r(P_n^1, P_n^1) = (n - 1)^2 + 1$. For $\ell > 1$, Balko-Cibulka-Král-Kynčl proved that $r(P_n^\ell, P_n^\ell) < c_\ell n^{128\ell}$ and asked for the growth rate for fixed ℓ . When $\ell = 2$, we improve this upper bound by proving $r(P_n^2, P_n^2) < cn^{19.5}$. Using this result, we determine the correct tower growth rate of the k -uniform hypergraph ramsey number of a $(k + 1)$ -clique versus an ordered tight path. Finally, we consider an ordered version of the classical Erdős-Hajnal hypergraph ramsey problem, improve the tower height given by the trivial upper bound, and conjecture that this tower height is optimal.

1 Introduction

Let K_n be the complete graph on n vertices. An ordered path P_s is the graph whose vertices are ordered as $v_1 < \dots < v_s$ and its edges are $v_1 v_2, v_2 v_3, \dots, v_{s-1} v_s$. the ramsey number $r(P_s, P_n)$ is the minimum N such that every red/blue coloring of $\binom{[N]}{2}$ results in a red copy of P_s or a blue copy of P_n . the famous Erdős-Szekeres monotone subsequence theorem states that $r(P_s, P_n) = (s - 1)(n - 1) + 1$. this implies that for fixed s , the ramsey number $r(P_s, P_n)$ is a polynomial function in n (in fact a linear function). On the other hand, it is well-known that $r(n, n) = r(K_n, K_n)$ has exponential growth rate. We begin by considering the case of ordered graphs that are denser than paths but sparser than cliques.

Definition 1. Given $\ell \geq 1$, the ℓ th power P_s^ℓ of a path P_s has ordered vertex set $v_1 < \dots < v_s$ and edge set $\{v_i v_j : |i - j| \leq \ell\}$. The ramsey number $r(P_s^\ell, P_n^\ell)$ is the minimum N such that every red/blue coloring of $\binom{[N]}{2}$ results in a red copy of P_s^ℓ or a blue copy of P_n^ℓ .

Note that $P_s^1 = P_s$. Conlon-Fox-Lee-Sudakov [3] asked whether $r(P_n^\ell, P_n^\ell)$ is polynomial in n for every fixed $\ell \geq 1$. Actually, the problem in [3] is about the ramsey number of ordered graphs with

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bandwidth at most ℓ but P_n^ℓ contains all such graphs so an upper bound for P_n^ℓ provides an upper bound for the bandwidth problem. This question was answered by Balko-Cibulka-Král-Kynčl [1] who proved an upper bound $cn^{128\ell}$ and asked (Problem 2 of [1]) for the growth rate of $r(P_n^\ell, P_n^\ell)$. The proof of our first result gives a slightly worse polynomial growth rate of $r(P_n^\ell, P_n^\ell)$ than in [1] for large ℓ but a much better one for small ℓ , in particular for $\ell = 2$. Note that the bound in [1] for $\ell = 2$ is cn^{256} .

Theorem 2. *There is an absolute constant $c > 0$ such that $r(P_n^2, P_n^2) < cn^{19.487}$ for all $n > 1$.*

Our second result is an application of Theorem 2 to a hypergraph ramsey problem. Indeed, this hypergraph ramsey problem is what motivated us to consider proving Theorem 2. Consider an *ordered* N -vertex k -uniform hypergraph H (k -graph for short), that is, a hypergraph whose vertex set is $[N] = \{1, 2, \dots, N\}$. A *tight path* of size s in H , denoted by P_s , comprises a set of s vertices $v_1, \dots, v_s \in [N]$, $v_1 < \dots < v_s$, such that $(v_j, v_{j+1}, \dots, v_{j+k-1}) \in E(H)$ for $j = 1, 2, \dots, s - k + 1$. the *length* of P_s is the number of edges, $s - k + 1$. Given (ordered) k -graphs F_1, F_2 , the Ramsey number $r_k(F_1, F_2)$ is the minimum N such that every red/blue coloring of the edges of the complete N -vertex k -graph K_N^k , whose vertex set is $[N] = \{1, \dots, N\}$, contains a red copy of F_1 or a blue copy of F_2 .

The famous cups-caps theorem of Erdős and Szekeres [8] implies that $r_3(P_s, P_n) = \binom{n+s-4}{s-2} + 1$. The author and Suk [11] considered the closely related problem of determining $r_k(P_s, n) := r_k(P_s, K_n^k)$ and showed that determining the tower height of $r_4(P_5, n)$ is equivalent to the notorious conjecture of Erdős-Hajnal and Rado on the tower height of $r_3(n, n)$. The results in [11] focused on fixed s and large n and there are no nontrivial results for the opposite case, namely for $r_k(s, P_n)$. Given the close connection between this problem and the problem of determining classical ramsey numbers, it would be of interest to obtain the growth rate in this range as well. Here we settle the first open case. Recall that the tower function $\text{twr}_i(x)$ is defined by $\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$.

Theorem 3. *For n large, $r_3(4, P_n) < n^{21}$ and more generally for each $k \geq 3$, there exists $c > 0$ such that for n large,*

$$\text{twr}_{k-2}(n^c) < r_k(k+1, P_n) < \text{twr}_{k-2}(n^{62}).$$

The main open problem here is to prove that $r_3(5, P_n)$ has polynomial growth rate and more generally that $r_3(s, P_n)$ has polynomial growth rate for all fixed $s \geq 4$. The corresponding results for higher uniformity follow easily from the case $k = 3$.

Our final topic considers a version of the well-known Erdős-Hajnal hypergraph ramsey problem with respect to tight paths. In order to shed more light on classical hypergraph ramsey numbers, Erdős and Hajnal [6] in 1972 considered the following more general parameter.

Definition 4. (Erdős-Hajnal [6]) For integers $2 \leq k < s < n$ and $2 \leq t \leq \binom{s}{k}$, let $r_k(s, t; n)$ be the minimum N such that every red/blue coloring of the edges of K_N^k results in a monochromatic blue copy of K_n^k or has a set of s vertices which contains at least t red edges.

Note that by definition $r_k(s, n) = r_k(s, \binom{s}{k}; n)$ so $r_k(s, t; n)$ includes classical ramsey numbers. The main conjecture of Erdős and Hajnal states that as t grows from 1 to $\binom{s}{k}$, there is a well-defined value $t_1 = h_1^{(k)}(s)$ at which $r_k(s, t_1 - 1; n)$ is polynomial in n while $r_k(s, t_1; n)$ is exponential in a

power of n , another well-defined value $t_2 = h_2^{(k)}(s)$ at which it changes from exponential to double exponential in a power of n and so on, and finally a well-defined value $t_{k-2} = h_{k-2}^{(k)}(s) < \binom{s}{k}$ at which it changes from twr_{k-2} to twr_{k-1} in a power of n . They were not able to offer a conjecture as to what $h_i^{(k)}(s)$ is in general, except when $i = 1$ (for which Erdős offered \$500) and when $s = k + 1$. For the latter, they conjectured that $h_i^{(k)}(k + 1) = i + 2$. This was solved for all but three i recently by the author and Suk [12] via the following result.

Theorem 5. (Mubayi-Suk [12]) *For $4 \leq t \leq k - 2$, there are positive $c = c(k, t)$ and $c' = c'(k, t)$ such that*

$$\text{twr}_{t-1}(c'n^{k-t+1} \log n) \geq r_k(k + 1, t; n) \geq \begin{cases} \text{twr}_{t-1}(cn^{k-t+1}) & \text{if } k - t \text{ is even} \\ \text{twr}_{t-1}(cn^{(k-t+1)/2}) & \text{if } k - t \text{ is odd.} \end{cases}$$

Here we consider the very same problem in the ordered setting by replacing K_n^k with P_n .

Definition 6. For integers $2 \leq k < s < n$ and $2 \leq t \leq \binom{s}{k}$, let $r_k(s, t; P_n)$ be the minimum N such that every red/blue coloring of the k -sets of $[N]$ results in a monochromatic blue copy of P_n or has a set of s vertices which contains at least t red edges.

Of course, $r_k(s, \binom{s}{k}, P_n) = r_k(s, P_n)$. We will focus our attention on the smallest case $s = k + 1$. Our main contribution here is the following conjecture which parallels the Erdős-Hajnal conjecture for cliques.

Conjecture 7. *For $3 \leq t \leq k$, there are positive $c = c(k, t)$ and $c' = c'(k, t)$ such that*

$$\text{twr}_{t-2}(n^c) < r_k(k + 1, t; P_n) < \text{twr}_{t-2}(n^{c'}).$$

This conjecture seems more difficult than the original problem of Erdős and Hajnal. For over 40 years the gaps in the bounds for the Erdős-Hajnal problem were between exponential and tower functions though now Theorem 5 shows that the correct growth rate is a tower function. For Conjecture 7 the gap is again between an exponential and a tower function but unfortunately the constructions used for Theorem 5 fail.

Using standard arguments, it is easy to prove an upper bound of the form $\text{twr}_{t-1}(n^c)$ in Conjecture 7 (see [12] for details). We improve this upper bound to the tower height given by Conjecture 7.

Theorem 8. *For all $3 \leq t \leq k$ there exists $c = c(k)$ such that $r_k(k + 1, t; P_n) < \text{twr}_{t-2}(cn^{2k})$.*

We make some further modest progress towards Conjecture 7 in the cases $t = 3$ and $(k, t) = (4, 4)$ by proving sharper bounds. Note that Theorem 3 determines the correct tower height of $r_k(k + 1, t; P_n)$ when $t = k + 1$; it is $k - 2 = t - 3$ so the formula differs from that in Conjecture 7.

Theorem 9. *The following bounds hold:*

- $r_k(k + 1, 3; P_n) \leq 16n$ for all $n \geq k \geq 3$
- $2n - 2 \leq r_3(4, 3; P_n) \leq 3n - 4$ for all $n \geq 3$
- $r_4(5, 4; P_n) \geq 2^n$.

We are not ready to offer a conjecture about the tower growth rate of $r_k(s, t; P_n)$ as t grows from 2 to $\binom{s}{k}$ for $s > k + 1$.

2 Proof of Theorem 2

We require bounds on the ramsey multiplicity problem introduced by Erdős [5]. This problem asks for the largest $\alpha = \alpha(\ell)$ such that every 2-edge coloring of K_N yields at least $(\alpha - o(1))N^\ell$ monochromatic copies of K_ℓ . Erdős [5] observed that $\alpha(\ell) > 0$ for all $\ell \geq 4$. The best known bounds for $\alpha(\ell)$ for large ℓ are very far apart and can be found in [2, 15]. We will use the specific result $\alpha(4) > 0.0287/4! = 0.0011958\bar{3}$ that was recently proved using Flag Algebras in [13] (see also [14]). Note that $\log_2(1/\alpha(4)) < 9.7434$.

Proof of Theorem 2. Let $\alpha = \alpha(4)$ be the constant from the Ramsey multiplicity problem above. Let $\epsilon = 10^{-9}$ and choose c such that every red/blue coloring of $\binom{[N]}{2}$ for $N > c$ results in at least $(\alpha - \epsilon)N^4$ monochromatic copies of K_4 . We will prove that

$$r(P_a^2, P_b^2) \leq c(ab)^{9.7435}$$

for all $a, b \geq 2$. This immediately gives the bounds we seek by letting $a = b = n$.

We will proceed by induction on $a + b$. If $a = 2$ then the trivial upper bound is $b < c(2b)^{9.7435}$ and the same holds if $b = 2$. For the induction step, suppose we have a red/blue coloring of $\binom{[N]}{2}$ with $N = c(ab)^{9.7435} > c$. By the definition of c , we obtain at least $(\alpha - \epsilon)N^4$ monochromatic copies of K_4 . Assume without loss of generality that half of these copies are red. To each such copy with vertex set $x_0 < x_1 < x_2 < x_3$ associate the middle pair of vertices x_1, x_2 . By the pigeonhole principle there exists a set $Y = y_1 < y_2$ that is the middle pair for at least $((\alpha - \epsilon)/2)N^4 / \binom{N-2}{2} > (\alpha - 2\epsilon)N^2 = \beta N^2$ red copies of K_4 . Let L be the set of smallest vertices y_0 in these red K_4 s and let R be the set of largest vertices y_3 in these red K_4 s. Note that all edges between Y and $L \cup R$ are red and that $|L||R|$ is the number of the red copies of K_4 that we are working with. Also both L and R have size at least βN for otherwise $\beta(1 - \beta)N^2 \geq |L||R| \geq \beta N^2$. By induction and $\log_2(1/\alpha) < 9.7434$,

$$|L| \geq \beta N = \beta c(ab)^{9.7435} > c(1/2)^{9.7435} (ab)^{9.7435} \geq r(P_{\lfloor a/2 \rfloor}^2, P_b^2).$$

Now we apply the induction hypothesis to find either a red $P_{\lfloor a/2 \rfloor}^2$ or a blue P_b^2 in L . If the latter occurs we are done so we get the former. The same argument applies to R . Consequently, we obtain two red copies of $P_{\lfloor a/2 \rfloor}^2$ one in L and the other in R . Consider these two copies together with Y . Since the distance between vertices in L and vertices in R is at least 3, and Y is connected to all vertices in $L \cup R$ by red edges, this yields a red copy of $P_{2\lfloor a/2 \rfloor + 2}^2$ which contains a red copy of P_a^2 because $2\lfloor a/2 \rfloor + 2 \geq a$. \square

3 Proof of Theorem 3

Definition 10. The (ordered) broom $B_{a,m}^k$ is the k -graph with vertices $v_1 < v_2 < \dots < v_a < w_1 < \dots < w_m$ such that v_1, \dots, v_a is a tight k -graph path and we also have all the edges $v_{a-k+2} \dots v_a w_j$ for all $j \in [m]$.

We will omit the superscript k in $B_{a,m}^k$ in all future usage as it will be obvious from the context. For example, in the proof below $k = 3$.

Theorem 11. $r_3(4, P_n) < 6nm$ where $m = r(P_n^2, P_n^2)$.

Proof. Recall that we are using the notation P_n for a 3-uniform tight path and P_n^2 for the square of a 2-uniform (i.e. graph) ordered path. We will prove that every red/blue coloring χ of $\binom{[N]}{3}$, where $N = 6nm$ yields either a red K_4^3 , a blue P_n , or a blue $B_{a,m}$ for all $2 \leq a \leq n$ within the first $6am$ vertices. Since $P_n \subset B_{n,m}$ this will prove the result. Suppose that there is no red K_4^3 or blue P_n . Let us show that there is a blue $B_{a,m}$ in $[6am]$ by induction on a . For the base case $a = 2$, we seek a pair of vertices $v_1 < v_2$ and at least m vertices $w > v_2$ within $[12m]$ such that $v_1 v_2 w$ is blue. If we cannot find these m vertices for any pair v_1, v_2 , then the number of blue edges is at most $\binom{12m}{2}(m-2) \leq 1/4 \binom{12m}{3}$ so the number of red edges is at least $(3/4) \binom{12m}{3}$ and a simple averaging argument shows that we would have a red K_4^3 , contradiction.

Now for the induction step, assume that we have a blue copy of $B_{a-1,m}$ in $[6(a-1)m]$ and we wish to augment this to a blue copy of $B_{a,m}$ in $[6am]$. Suppose that the vertex set of the blue $B_{a-1,m}$ is

$$v_1 < v_2 < \dots < v_{a-1} < w_1 < \dots < w_m.$$

Define the red/blue coloring ϕ of the complete graph on $\{w_1, \dots, w_m\}$ by $\phi(w_i w_j) = \chi(v_{a-1} w_i w_j)$. By definition of m , we get a copy H of a monochromatic P_n^2 under ϕ with vertices $z_1 < \dots < z_n$.

Suppose H is red under ϕ . The four vertices $v_{a-1}, z_i, z_{i+1}, z_{i+2}$ have three red edges, so $\chi(z_i z_{i+1} z_{i+2})$ is blue for all i . We conclude that $z_1 < \dots < z_n$ is a blue P_n .

Next, suppose H is blue under ϕ . Fix i and consider the three vertices z_i, z_{i+1}, z_{i+2} . If there are at least m edges $z_i z_{i+1} y$ with $y \leq 6am$ and $\chi(z_i z_{i+1} y)$ is blue, then we can use these edges to form a blue copy of $B_{a,m}$ in $[6am]$ with vertices

$$v_2 < \dots < v_{a-1} < z_i < z_{i+1} < Y$$

where Y is the set of these y . So the number of such y is at most m , and the same is true for the pairs z_{i+1}, z_{i+2} and z_i, z_{i+2} . Since $6am - 6(a-1)m = 6m > 3m$ there is a vertex y such that $\chi(z_i z_{i+1} y) = \chi(z_i z_{i+2} y) = \chi(z_{i+1} z_{i+2} y)$ and these are all red. Therefore $\chi(z_i z_{i+1} z_{i+2})$ is blue. Since this argument applies for each i , we obtain a blue P_n under χ with vertices $z_1 < \dots < z_n$. \square

Proof of Theorem 3. The case $k = 3$ of Theorem 3 follows immediately from Theorem 2 and Theorem 11. The lower bound for general k follows from the lower bound for $r_k(P_{k+1}, P_n)$ in [9, 10] (see also [4]). The upper bound for general k follows from the upper bound when $k = 3$ (as a base case) and the standard pigeonhole argument for hypergraph ramsey numbers due to Erdős, Hajnal and Rado [7]. Applying this argument from $k = 3$ to $k = 4$ raises the exponent of n from slightly less than 20.5 to slightly less than 61.5 and subsequent applications do not affect the exponent of n . \square

4 Proof of Theorem 8

Let $f_k(n)$ be the minimum N such that every red/blue coloring of $\binom{[N]}{k}$ results in a blue P_n or a set S of $k+1$ vertices with at least 3 red edges, one of which consists of the smallest k vertices in S . Let $H_k(3)$ denote the set of ordered k -graphs with three red edges as described above. We will abuse notation by saying a copy of $H_k(3)$ when we mean a copy of some $H \in H_k(3)$.

Theorem 12. $f_k(n) \leq 2n^2$ for all $n > k \geq 3$.

Proof. We will prove that every red/blue coloring χ of $\binom{[N]}{k}$, where $N = 2n^2$ yields either a red $H_k(3)$ or a blue P_n or a blue $B_{n-1,n}$. Since $P_n \subset B_{n-1,n}$ this will prove the result. Assume that there is no red $H_k(3)$ and no blue P_n . We will prove that there is a blue $B_{n-1,n}$ by showing that there is a blue $B_{a,n}$ in $[2an]$ for all $1 \leq a \leq n-1$ by induction on a . The base case $a = 1$ is trivial since $B_{1,n}$ is just a collection of $n+1$ vertices (with no edge since $k \geq 3$) and $2n > n+1$.

Now for the induction step, assume that we have a blue copy B of $B_{a-1,n}$ in $[2(a-1)n]$ and we wish to augment this to a blue copy of $B_{a,n}$ in $[2an]$. Suppose that the vertex set of the blue $B_{a-1,n}$ is

$$v_1 < v_2 < \dots < v_{a-1} < w_1 < \dots < w_n.$$

Let $V = \{v_1, \dots, v_{a-1}\}$ and $W = \{w_1, \dots, w_n\}$. For each $0 \leq j \leq k$, let $S_j = \{v_{a-j}, \dots, v_{a-1}\}$ denote the j largest vertices of V .

Claim. For every $i \in [k]$ and $P \in \binom{W}{i}$, $\chi(S_{k-i} \cup P)$ is blue.

Proof of Claim. Let us proceed by induction on i . The base case $i = 1$ is trivial, due to the definition of B . Indeed, if $a-1 \geq k-1$, then all k -sets of the form $S_{k-1} \cup \{w\}$ for $w \in W$ are blue. If $a-1 < k-1$ there is nothing to check. For the induction step, let $i \geq 2$ and suppose for contradiction that $\chi(S_{k-i} \cup P)$ is red for some $P \in \binom{W}{i}$. Let P_1, P_2 be two distinct $(i-1)$ -sets contained in P . Note that $i \geq 2$ means that $\binom{i}{i-1} \geq 2$ so such P_1, P_2 exist. Suppose that there are n vertices $w'_1 < w'_2 < \dots < w'_n \leq 2an$ such that $w'_1 > w_n$ and $\chi(S_{k-i} \cup P_1 \cup \{w'_i\})$ is blue. Then $V \cup P_1 \cup \{w'_1, \dots, w'_n\}$ is a blue $B_{a+i-2,n}$ which contains a blue $B_{a,n}$ in $[2an]$ as required. Indeed, it suffices to check that $\chi(S_{k-j} \cup Q_j)$ is blue for all $j \leq i-1$, where Q_j is the set of j smallest vertices of P_1 . But $Q_j \in \binom{W}{j}$ and $j < i$ so this is true by induction on i . We conclude that the number of such vertices w'_i is at most $n-1$ and the same assertion holds for P_1 replaced by P_2 . This gives at most $2n-2$ vertices between w_n and $2an$. Since $w_n \leq 2(a-1)n = 2an - 2n$, there exists a w such that $w_n < w \leq 2an$ such that both $\chi(S_{k-i} \cup P_1 \cup \{w\})$ and $\chi(S_{k-i} \cup P_2 \cup \{w\})$ are blue. This means that we have a red copy of $H_k(3)$ in $S_{k-i} \cup P \cup \{w\}$, contradiction.

Now we simply apply the Claim with $i = k$ to conclude that all k -sets of $\binom{W}{k}$ are blue, and this is a blue clique with n vertices which contains a blue P_n . \square

Proof of Theorem 8. Let $f_k(k+1, t; P_n)$ be the minimum N such that every red/blue coloring of $\binom{[N]}{k}$ results in a blue P_n or a set S of $k+1$ vertices with at least t red edges, one of which consists of the smallest k vertices in S . We observe that $f_k(k+1, t; P_n) < 2^{\binom{f_{k-1}(k, t-1; P_{n-1})}{k}}$. Indeed, this follows from the standard pigeonhole argument for hypergraph ramsey numbers due to Erdős,

Hajnal and Rado [7]. When applying this argument, we must note that the initial red edge in the $(k-1)$ -graph gives rise to two red edges in the k -graph, one of which is again an initial edge. We apply this recurrence repeatedly until we have a $(k-t+3)$ -graph. Theorem 12 now applies to give an upper bound $2n^2$ and this yields the result. \square

5 Proof of Theorem 9

Let us first prove the $t = 3$ case of Theorem 9. Let $H(3)$ be the collection of ordered k -graphs with $k+1$ vertices and at least three edges.

Proof of Theorem 9 for $t = 3$. We are to show that $r_k(k+1, 3; P_n) \leq 16n$ for all $n \geq k \geq 3$. We will prove that every red/blue coloring χ of $\binom{[N]}{3}$, where $N = 16n$ yields either a red $H \in H(3)$, a blue P_n , or a blue $B_{a,6}$ for all $k-1 \leq a < n$ within the first $16(a+1)$ vertices. As $P_n \subset B_{n-1,6}$ this will prove the result. Suppose that there is no red $H \in H(3)$ or blue P_n . Let us show that there is a blue $B_{a,6}$ in $[16(a+1)]$ by induction on a . For the base case $a = k-1$, we seek $k-1$ vertices $v_1 < \dots < v_{k-1}$ and at least 6 vertices $w > v_{k-1}$ within $[16k]$ such that $v_1 \dots v_{k-1}w$ is blue. If we cannot find these 6 vertices for any $(k-1)$ -set $v_1 < \dots < v_{k-1}$, then the number of blue edges in $[16k]$ is at most $5\binom{16k}{k-1}$. A short calculation shows that this is less than $((k-1)/(k+1))\binom{16k}{k}$ since $k \geq 3$. Consequently, the number of red edges is more than $(2/(k+1))\binom{16k}{k}$ and an easy averaging argument then implies that there is a some red $H \in H(3)$

Now for the induction step, assume that we have a blue copy of $B_{a-1,6}$ in $[16a]$ and we wish to augment this to a blue copy of $B_{a,6}$ in $[16(a+1)]$. Suppose that the vertex set of the blue $B_{a-1,6}$ is

$$v_1 < v_2 < \dots < v_{a-1} < w_1 < \dots < w_6.$$

For $q \in \{k-1, k-2, k-3\}$, let $S_q = \{v_{a-q}, \dots, v_{a-1}\}$ be the greatest q vertices among the v_i . Define the red/blue coloring ϕ of the complete graph on $\{w_1, \dots, w_6\}$ by $\phi(w_i w_j) = \chi(S_{k-2} w_i w_j)$. Since $r(3, 3) \leq 6$, we obtain a monochromatic triangle T under ϕ . If T is red, then $T \cup S_{k-2}$ yields a red member of $H(3)$ and we are done, so T is blue. Assume for simplicity that $T = w_1 < w_2 < w_3$. Fix $1 \leq i < j \leq 3$. If there are at least 6 edges $S_{k-3} w_i w_j y$ with $w_3 < y \leq 16(a+1)$ and $\chi(S_{k-3} w_i w_j y)$ is blue, then we can use these edges to form a blue copy of $B_{a,m}$ in $[16(a+1)]$ with vertices

$$v_2 < \dots < v_{a-1} < w_i < w_j < Y$$

where Y is the set of these y . So the number of such y is at most 5, and the same is true for all three pairs $\{i, j\}$. Since $16(a+1) - 16a = 16 > 15$ there is a vertex y such that $\chi(S_{k-3} w_1 w_2 y) = \chi(S_{k-3} w_1 w_3 y) = \chi(S_{k-3} w_2 w_3 y)$ and these are all red. This gives a red member of $H(3)$. \square

Proof of Theorem 9 for $(k, t) = (4, 3)$. For the lower bound, consider the ordering of $2n-3$ points

$$v_1, v'_1 < v_2, v'_2 < \dots < v_{n-2}, v'_{n-2} < v_{n-1}.$$

For all $i < j$, color the triples $v_i v'_i v_j$ red. Color all other triples blue. No blue path P_q can contain both v_i and v'_i unless they appear at the end of P_q . If no such pair is in P_q then clearly $q \leq n-1$.

If v_i, v'_i are both in P_q , then they lie at the end of the path, and $q \leq (i-1) + 2 = i+1 \leq n-1$. Hence there is no blue P_n . If we have four points with two red edges, then two of these triples must be $v_i v'_i a$ and $v_i v'_i b$ for some $v_i < a, b$. But then there cannot be any other red edge among these points.

For the upper bound, define P'_q to be the ordered 3-graph obtained from P_q by adding a new vertex $v'_q > v_{q-1}$ and a new edge $v_{q-2} v_{q-1} v'_q$. think of v'_q as a clone of v_q . We will prove the following stronger statement by induction on n :

$$r_3(4, 3; P'_n) \leq 3n - 3.$$

Since $P_n \subset P'_n$ this will prove the upper bound. the base case $n = 3$ is true due to the following simple argument. Suppose we have a red/blue coloring of $\binom{[6]}{2}$. If there are two edges of the form $12i$ that are blue then we have a blue P'_3 so there is at most one such edge. this means that there are $2 < i < j < k \leq 6$ such that $12x$ is red for all $x \in S = \{i, j, k\}$. If any edge of the form $1xy$ or $2xy$ is red then for $x, y \in S$, then $\{1, 2, x, y\}$ has three red edges, so all such edges are blue. this means that $1ij$ and $1ik$ are both blue, giving a blue P'_3 .

Now for the induction step, assume that $N = 3n - 3$ and we have a red/blue coloring χ of the triples of $[N]$ with no four points containing three red edges. By induction, we obtain a blue P'_{n-1} in $[N - 3]$. Say the vertices of this P'_{n-1} are

$$v_1 < \dots < v_{n-2} < v_{n-1}, v'_{n-1}$$

and assume by symmetry that $v_{n-1} < v'_{n-1}$. Let $S = \{N - 2, N - 1, N\}$. If there are two blue edges of the form $v_{n-2} v_{n-1} x$ and $v_{n-2} v_{n-1} y$ for $x, y \in S$, then we have obtained a blue copy of P'_n with $x = v_n$ and $y = v'_n$. So at most one of these edges is blue. The same applies to $v_{n-2} v'_{n-1} x$ and $v_{n-2} v'_{n-1} y$. Since $|S| = 3$, there is $w \in S$ such that both $v_{n-2} v_{n-1} w$ and $v_{n-2} v'_{n-1} w$ are red. Now consider the four vertices $v_{n-2}, v_{n-1}, v'_{n-1}, w$. We have identified two red edges among them so $v_{n-2} v_{n-1} v'_{n-1}$ and $v_{n-1} v'_{n-1} w$ are blue. If there exists $z \in S - \{w\}$ such that $\chi(v_{n-1} v'_{n-1} z)$ is blue, then we obtain a copy of P'_n as follows:

$$v_2 < \dots < v_{n-2} < v_{n-1} < v'_{n-1} < w, z.$$

Therefore $\chi(v_{n-1} v'_{n-1} a) = \chi(v_{n-1} v'_{n-1} b)$ and both are red, where $S = \{a, b, w\}$. If $\chi(v_{n-2} v_{n-1} x)$ is blue for some $x \in \{a, b\}$, then we obtain the blue P'_n

$$v_1 < \dots < v_{n-2} < v_{n-1} < v'_{n-1}, x.$$

Hence $\chi(v_{n-2} v_{n-1} x)$ is red for $x \in \{a, b\}$. The four vertices $v_{n-2}, v_{n-1}, v'_{n-1}, x$ contain the two red edges $v_{n-2} v_{n-1} x$ and $v_{n-1} v'_{n-1} x$ so $\chi(v_{n-2} v'_{n-1} x)$ is blue for $x \in \{a, b\}$. this gives us

$$v_1 < \dots < v_{n-2} < v'_{n-1} < a, b$$

which is a blue P'_n , and the proof of $r_3(4, 3; P'_n) \leq 3n - 3$ is complete.

Observe that $r_3(4, 3; P_n) \leq r_3(4, 3; P'_n) - 1$ by taking an optimal construction for $r_3(4, 3; P_n)$, adding a new largest vertex v and coloring all triples containing v with blue. \square

Proof of Theorem 9 for $(k, t) = (4, 4)$. We are to show that $r_4(5, 4; P_n) \geq 2^n$. Proceed by induction on n . The case $n = 1$ is trivial, so assume we have a construction for $n - 1$ that uses the vertices $[2^{n-1}]$. To obtain the construction for n , take a copy of the construction for $n - 1$ among the vertices $\{2^{n-1} + 1, \dots, 2^n\}$. It remains to color 4-sets that intersect both halves of $[2^n]$. We color all the 4-sets that have exactly two points in each half red and all other 4-sets blue. Let us first argue that no 5 points contain 4 red edges. If all 5 points lie in one half then we are done by induction. If the distribution of points is $4 + 1$ then we again have at most one red edge and if the distribution is $3 + 2$ then we have exactly three red edges. The other cases are of course symmetric. Next we argue that there is no blue P_n . Such a blue P_n cannot have two points in both halves as the $2 + 2$ edges are red so all but one point must lie in one half. This gives a blue P_{n-1} in one half which cannot exist by induction. \square

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